BOUNDARY LAYER CALCULATION ON THE INSULATED WALLS OF A MHD GENERATOR CHANNEL WITH ALLOWANCE FOR HALL CURRENT

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An examination is made of the laminar boundary layer of low-temperature plasma on the insulated walls of a MHD generator channel with arbitrary magnetic field along the channel, perpendicular to the wall. An example of a boundary layer calculation by this method is given.

When an electrically conducting, low-temperature plasma flows in a MHD generator channel, the boundary layers formed near the walls and the electrodes have an appreciable influence on generator characteristics.

The present paper examines the laminar boundary layer on the insulated walls with allowance for Hall current. The method of calculation described enables some of the poorly founded assumptions used by Kerrebrock [1] to be avoided, particularly the assumption of local similarity.

1. Statement of the problem. A stream of low-temperature plasma moves in a rectangular channel with velocity u (Fig. 1). Two of the walls are insulated, the other two being electrodes. Motion occurs at right angles to the magnetic field, which has an arbitrary distribution along the channel. The electric current generated has two components: a current j_y directed across the channel, and a Hall current j_x along the channel axis. As a result of the Hall current, there is a transverse velocity component v in the boundary layer, causing appreciable three-dimensional effects in the boundary layer on the insulated walls.

The solution is carried out under the following assumptions:

1. The plasma stream is steady.
2. The flow is laminar.
3. The plasma is compressible.
4. $Re_m \ll 1$. The magnetic field has one component along the z axis, and this is a function of x.
5. The Hall coefficient is $\beta_e = \omega_e \tau_e \neq 0$.

The equations describing the boundary layer may be simplified appreciably by taking into account the fact that the transverse velocity component v due to the Hall current is not large, and by omitting terms of order δ and less in the equations, the boundary layer equations may be written as follows:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) - j_y B, \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z}\right) = \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) + j_x B, \tag{3}$$

$$\rho u\frac{\partial}{\partial x}\left[C_p T + \frac{u^2+v^2}{2}\right] + \rho w\frac{\partial}{\partial z}\left[C_p T + \frac{u^2+v^2}{2}\right] = \frac{\partial}{\partial z}\left[\lambda\frac{\partial T}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial}{\partial z}\left(\frac{u^2+v^2}{2}\right)\right] + j_x E_x - j_y E_y, \tag{4}$$

$$p = \rho g R T, \tag{5}$$

$$\mu = \mu_*(T/T_*)^n, \quad \lambda = \lambda_*(T/T_*)^n, \tag{6}$$

$$\sigma = \sigma(p, T) \tag{7}$$

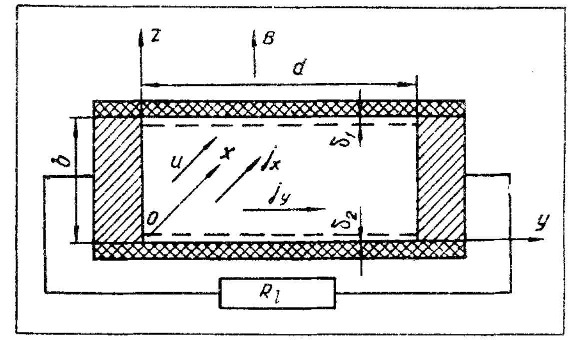

Fig. 1. Cross section of the MHD generator channel.

with boundary conditions

$$u = v = w = 0, \quad T = T_w \text{ when } z = 0,$$
$$u \to u_\infty(x), \quad w = v \to 0, \quad T \to T_\infty \text{ when } z \to \infty. \tag{8}$$

The system of equations obtained coincides with that used in Kerrebrock's analysis [1]. We have managed to eliminate v from (3), and we may therefore examine the problem as a plane one, using the stream function ψ.

From the solution of the problem, in the one-dimensional approximation, of the motion of an inviscid non-heat-conducting plasma in crossed electric and magnetic fields, under the condition T_∞ = const, the flow parameters $p_\infty(x)$, $u_\infty(x)$, $j_\infty(x)$ in the stream core are determined. We consider that there is no Hall current in the core, which is possible where there is an applied electric field along the flow axis

$$E_x = \beta_{e\infty}(E_y - u_\infty B). \tag{9}$$

We transform Eqs. (2)-(4) with the help of the stream core equations

$$\frac{dp}{dx} = -j_\infty B - \rho_\infty u_\infty \frac{du_\infty}{dx}, \tag{10}$$

$$\rho_\infty u_\infty^2 \frac{du_\infty}{dx} = j_\infty E_y. \tag{11}$$

Substituting (10) and (11) into (2)–(4), and assuming n = 1 in (6), we obtain

$$\rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} =$$

$$\rho_\infty u_\infty \frac{du_\infty}{dx} + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) + B(j_\infty - j_y), \tag{2'}$$

$$\rho u \frac{\partial v}{\partial x} + \rho w \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) + j_x B, \tag{3'}$$

$$\rho u \frac{\partial}{\partial x}\left[C_p T + \frac{u^2 + v^2}{2}\right] + \rho w \frac{\partial}{\partial z}\left[C_p T + \frac{u^2 + v^2}{2}\right] =$$

$$= \frac{\partial}{\partial z}\left[\lambda \frac{\partial T}{\partial z}\right] + \frac{\partial}{\partial z}\left[\mu \frac{\partial}{\partial z}\left(\frac{u^2+v^2}{2}\right)\right] + j_x E_x -$$

$$-\rho_\infty u_\infty^2 \frac{du_\infty}{dx} \frac{j_y}{j_\infty}. \tag{4'}$$

2. Method of solution. The solution of (2')–(4') is effected by introducing into the equations the Dorodnitsyn variables [1, 3]

$$\bar{x} = \int_0^x \frac{p}{p_*} dx, \quad \eta = \left(\frac{u_*}{2\nu_* \bar{x}}\right)^{1/2} \int_0^z \frac{\rho}{\rho_*} dz. \tag{12}$$

Taking into account the experimental data obtained in flow of low-temperature plasmas in rectangular channels, and also the fact that the optimum construction for generators is a channel in the shape of a diffuser, the relation $u_\infty(\bar{x})$ may be approximated with a high degree of accuracy by the "one-slope" velocity profile formula

$$u_\infty(\bar{x}) = u_*(1 - m\bar{x}) = u_*(1 - \xi), \tag{13}$$

where $\xi \ll 1$.

Let us introduce the stream function ψ, satisfying the continuity equation (1)

$$\rho u = \frac{\partial \psi}{\partial z}, \quad \rho w = -\frac{\partial \psi}{\partial x}. \tag{14}$$

Following Howarth* [2], we shall represent the stream function as a power series in ξ:

$$\psi = \sqrt{2u_* \nu_* \bar{x}}\,[f_0(\eta) - (8\xi) f_1(\eta) + (8\xi)^2 f_2(\eta) - \ldots]. \tag{15}$$

Similarly, we shall expand the dimensionless temperature Θ in a series of ascending powers of ξ:

$$\Theta = T/T_\infty = \Theta_0(\eta) - (8\xi)\Theta_1(\eta) + (8\xi)^2\Theta_2(\eta) - \ldots \tag{16}$$

The stream velocities in the longitudinal and transverse directions may be written in the form

$$\bar{u} = u/u_* = \frac{1}{2}[f_0'(\eta) -$$

$$-(8\xi) f_1'(\eta) + (8\xi)^2 f_2'(\eta) - \ldots], \tag{17)**$$

$$\bar{v} = v/u_* = [-(8\xi) r_1(\eta) + (8\xi)^2 r_2(\eta) - \ldots]. \tag{18}$$

Auxiliary equations [1] were used in the solution:

for the electrical conductivity through the boundary layer

$$\sigma = \sigma_\infty \exp\{-\chi[1/\Theta - 1]\} \tag{19}$$

(where $\chi = \varphi/2k$, φ is the ionization potential of the relevant alkali metal);

for the Hall coefficient under equilibrium conduction (19)

$$\beta_e = \beta_{e\infty} \Theta^{1/2}. \tag{20}$$

*In calculating the boundary layers the Howarth method was used, first applied, as far as we know, to flow of compressible fluids in a magnetic field by Genkin [5].

**Here and below the prime denotes differentiation with respect to η.

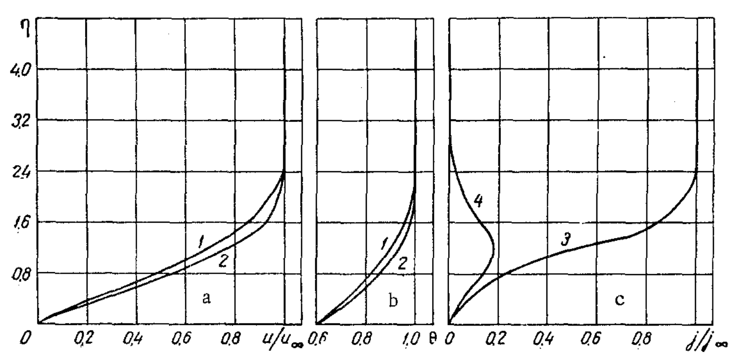

Fig. 2. Dependence of a) u/u_∞, b) Θ, and c) j/j_∞ on η in the boundary layer on an insulated wall in a MHD generator channel ($x = 0.8$; $K = 0.5$; $\beta_{e\infty} = 2.5$; $M_* = 1.0$): 1) without allowing for current terms; 2) allowing for j_y and j_x; 3) distribution of the transverse electric current j_y/j_∞; 4) distribution of Hall current j_x/j_∞.

The dimensionless electric currents are found from the expressions

$$\frac{j_y}{j_\infty}=\frac{\sigma}{\sigma_\infty}\left(\frac{1}{1+\beta_{e^2}}\right)\Big[\left(\frac{K}{K-1}\right)+ + \beta_e\beta_{e\infty}-\left(\frac{1}{K-1}\right)\left(\frac{u}{u_\infty}-\beta_e\frac{v}{u_\infty}\right)\Big], \quad (21)$$

$$\frac{j_x}{j_\infty}=\frac{\sigma}{\sigma_\infty}\left(\frac{1}{1+\beta_{e^2}}\right)\Big[\beta_{e\infty}-\left(\frac{K}{K-1}\right)\beta_e+ +\left(\frac{1}{K-1}\right)\left(\beta_e\frac{u}{u_\infty}+\frac{v}{u_\infty}\right)\Big], \quad (22)$$

where $K = E_y/u_\infty B$ is the electromagnetic load factor, and

$$j_\infty=\sigma_\infty(E_y-u_\infty B). \quad (23)$$

Following substitution of all the relations used into (2')-(4'), and slight transformations, we obtain three infinite systems of ordinary differential equations by equating coefficients of identical powers of ξ.

1. For the equations of motion in the longitudinal direction (2')

$$f_0'''+f_0f_0''=0, \quad (24)$$

$$f_1'''+f_0f_1''-2f_0'f_1'+3f_0''f_1=-\Theta_0- -\frac{\Theta_0}{K}\Big\{1-\Pi\Big[\left(\frac{K}{K-1}\right)+ +\beta_{e\infty}^2\Theta_0^{1/2}+\left(\frac{1}{K-1}\right)\frac{f_0'}{2}\Big]\Big\}, \quad (25)$$

$$f_2'''+f_0f_2''-4f_0'f_2'+5f_0''f_2=2f_1'^2-3f_1f_1''- -\left(\Theta_1+\frac{\Theta_0}{8}\right)\left(1+\frac{1}{K}\right)+\frac{\Pi}{K}\Big[\left(\frac{K}{K-1}\right)+\beta_{e\infty}^2\Theta_0^{1/2}\Big]\times \times\left(\Theta_1+\frac{1}{8}\Theta_0\right)-\left(\frac{1}{K-1}\right)\Big[\frac{f_0'}{2}\left(\Theta_1+\frac{\Theta_0}{8}\right)+ +\Theta_0\left(\frac{f_1'}{2}-\frac{f_0'}{16}-\beta_{e\infty}\Theta_0^{1/2}r_1\right)\Big], \quad (26)$$

. . . .

2. For the equations of motion in the transverse direction (3')

$$r_1''+f_0r_1'-2r_1f_0'=-\frac{\Pi}{2K}\Big\{\Big[\beta_{e\infty}-\left(\frac{K}{K-1}\right)\beta_{e\infty}\Theta_0^{1/2}\Big]\Theta_0+ +\left(\frac{1}{K-1}\right)\frac{\beta_{e\infty}}{2}f_0'\Theta_0^{3/2}\Big\}, \quad (27)$$

$$r_2''+f_0r_2'-4f_0'r_2=2r_1f_1'-3f_1r_1'- -\frac{\Pi}{2K}\Big\{\Big[\beta_{e\infty}-\left(\frac{K}{K-1}\right)\beta_{e\infty}\Theta_0^{1/2}\Big]\left(\Theta_1+\frac{\Theta_0}{8}\right)+ +\left(\frac{1}{K-1}\right)\Big[\frac{\beta_{e\infty}}{2}f_0'\Theta_0^{1/2}\left(\Theta_1+\frac{\Theta_0}{8}\right)\Big]\Big\}+ +\frac{\Pi}{2K}\left(\frac{1}{K-1}\right)\Big[-\beta_{e\infty}\Theta_0^{1/2}\left(\frac{f_1'}{2}-\frac{f_0'}{16}\right)-r_1\Big]\Theta_0, \quad (28)$$

. . . .

3. For the energy equations (4')

$$\frac{1}{\mathrm{Pr}_*}\Theta_0''+f_0\Theta_0'=-\beta_1f_0''^2, \quad (29)$$

$$\frac{1}{\mathrm{Pr}_*}\Theta_1''+f_0\Theta_1'-2f_0'\Theta_1=-3\Theta_0'f_1- -\beta_1[f_0f_0'f_1''+3f_0'f_0''f_1+f_0f_1'f_0''-2f_0''^2f_1'+ +2f_0''f_1''+f_1'f_0'''+f_1'''f_0']- -2\beta_1\Gamma\Big\{\Big[\beta_{e\infty}-\left(\frac{K}{K-1}\right)\beta_{e\infty}\Theta_0^{1/2}\Big]\Theta_0+ +\left(\frac{1}{K-1}\right)\frac{\beta_{e\infty}}{2}f_0'\Theta_0^{3/2}\Big\}+2\beta_1\Pi\times \times\Big\{\Big[\left(\frac{K}{K-1}\right)+\beta_{e\infty}^2\Theta_0^{1/2}\Big)\Theta_0-\left(\frac{1}{K-1}\right)\frac{f_0'}{2}\Theta_0\Big\}, \quad (30)$$

$$\frac{1}{\mathrm{Pr}_*}\Theta_2''+f_0\Theta_2'-4f_0'\Theta_2=2\Theta_1f_1'-3\Theta_1'f_1-5\Theta_0'f_2+ +\beta_1[4f_0'^2f_2'+4f_0'f_1'^2-5f_0'f_0''f_2-3f_0'f_1f_1''- -f_0f_0'f_2''-f_0f_1'f_1''-3f_0''f_1f_1'-f_0f_0''f_2'-f_2'''f_0'- -f_1'f_1'''-f_1''^2-2f_0''f_2''-f_0'''f_2']+ +4\beta_1[2r_1^2f_0'-r_1r_1'f_0-r_1'^2-r_1r_1'']- -2\beta_1\Gamma\Big\{\Big[\beta_{e\infty}-\left(\frac{K}{K-1}\right)\beta_{e\infty}\Theta_0^{1/2}\Big]\left(\Theta_1+\frac{1}{4}\Theta_0\right)+ +\left(\frac{1}{K-1}\right)\frac{\beta_{e\infty}}{2}f_0'\Theta_0^{1/2}\left(\Theta_1+\frac{\Theta_0}{4}\right)\Big\}+ +2\beta_1\Gamma\left(\frac{1}{K-1}\right)\Big[-\beta_{e\infty}\Theta_0^{1/2}\frac{f_1'}{2}+ +\frac{1}{16}f_0'\beta_{e\infty}\Theta_0^{1/2}-r_1\Big]\Theta_0+ +2\beta_1\Pi\Big\{\Big[\left(\frac{K}{K-1}\right)+\beta_{e\infty}^2\Theta_0^{1/2}\Big]\times \times\left(\Theta_1+\frac{\Theta_0}{4}\right)-\left(\frac{1}{K-1}\right)\frac{f_0'}{2}\times \times\left(\Theta_1+\frac{\Theta_0}{4}\right)\Big\}+2\beta_1\Pi\left(\frac{1}{K-1}\right)\times \times\left(-\frac{f_1'}{2}+\frac{f_0'}{16}+\beta_{e\infty}\Theta_0^{1/2}r_1\right)\Theta_0, \quad (31)$$

. . . .

These systems are solved crosswise. Because of the smallness of ξ we may confine attention to the second approximation.

In the equations we have put

$$\Pi=[1/(1+\beta_{e\infty}^2\Theta_0)]\exp\{-\chi[1/\Theta_0-1]\}, \quad (32)$$

$$\Gamma=\beta_{e\infty}[1/(1+\beta_{e\infty}^2\Theta_0)](1-1/K)\exp\{-\chi[1/\Theta_0-1]\}, \quad (33)$$

$$\beta_1=(\gamma-1)(M_*^2/4).$$

For simplification in (24)-(31) we have used

$$\sigma=\sigma_\infty\exp\{-\chi[1/\Theta_0-1]\}, \quad (34)$$

$$\beta_e=\beta_{e\infty}\Theta_0^{1/2}. \quad (35)$$

For calculation purposes the results obtained may be improved using (19) and (20).

To obtain the solutions the wall temperature Θ_w was assumed to be constant, while the electromagnetic load factor K may be constant or may be assigned in the form of a function of channel length.

The boundary conditions which must be satisfied by the functions

$$f_0,\ f_1,\ f_2,\ \ldots,\ f_n,\ r_1,\ r_2,\ \ldots,$$
$$r_n,\ \Theta_0,\ \Theta_1,\ \Theta_2,\ \ldots,\ \Theta_n,$$

are derived from the general boundary conditions (8),

when $\eta = 0$

$$f_0 = f_1 = f_2 = \ldots = f_n = 0,$$
$$f_0' = f_1' = f_2' = \ldots = f_n' = 0,$$
$$r_1 = r_2 = \ldots = r_n = 0,$$
$$\Theta_0 = \Theta_w,\ \Theta_1 = \Theta_2 = \ldots = \Theta_n = 0;$$

when $\eta \to \infty$

$$f_0' \to 2.0,\ f_1' \to 0.25,\ f_2' = \ldots = f_n' \to 0,$$
$$r_1 = r_2 = \ldots = r_n \to 0,$$
$$\Theta_0 \to 1.0,\ \Theta_1 = \Theta_2 = \ldots = \Theta_n \to 0.$$

3. Example of calculation. Using the above equations, we calculated the thermodynamic, hydrodynamic, and electrical parameters of a plasma in the boundary layer on an insulated wall in a MHD generator channel with the following initial data: the plasma is helium with 1.5% by weight of cesium; $p_* = 1.22$ atm. abs; $T_\infty = T_* = 2400°$ K; m = 0.1; B = 2.0 Wb/m^2; $u_* = 2800$ m/sec; L = 1.0 m; b = $2.2 \cdot 10^{-2}$ m, d = $5.0 \cdot 10^{-2}$ m; $\chi = 7.5$; K = 0.5; $T_w = 1440°$ K; $Pr_* = 0.7$.

Figure 2 shows $\bar{u}$, Θ, j_y, $\bar{j}_x$ for the section $\bar{x} = 0.8$. Comparison of the value for plasma longitudinal velocity $\bar{u}$ and temperature Θ with the values in the ordinary thermal boundary layer, indicates that there is negligible retardation of the plasma due to electromagnetic forces, negligible redistribution of velocity profile in the boundary layer, and negligible temperature increase due to joule heating of the plasma.

The velocity gradients normal to the wall and the temperatures are increased, leading to increased friction and heat transfer on the insulated walls. The relevant expressions may be easily obtained [3, 4].

The transverse velocity v due to the Hall effect is insignificant because of the weak magnetohydrodynamic interaction.

It should be noted that the choice of electromagnetic load factor K and wall temperature Θ_w has a considerable influence on the distribution of the plasma parameters in the boundary layer.

NOTATION

V(u, v, w)—plasma velocity; ρ—plasma density; μ—dynamic viscosity; ν—kinematic viscosity; Re_m—magnetic Reynolds number; R—gas constant; p—pressure; C_p—plasma specific heat; λ—thermal conductivity; T—stream temperature; g—free-fall acceleration; B—magnetic induction; j—electric current density; σ—electrical conductivity; β_e—Hall coefficient; E—electric field intensity; K—electromagnetic load factor; k—Boltzmann constant; χ—coefficient; ψ—stream function; δ—magnetodynamic boundary layer thickness; L, b, and d—length, height, and width of channel; γ—adiabatic exponent; M—Mach number. Subscripts: *—conditions at initial section; ∞—in flow core; ω—at wall.

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